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Publication date:
1975

Document Version
Publisher's PDF, also known as Version of record

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Research memorandum

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INVARIANT COMPETITIVE EQUILIBRIUM IN A DYNAMIC ECONOMY WITH NEGOTIABLE SHARES.*

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* Revised version of Cowles Foundation Discussion Paper No 401, August 1975.

INVARIANT COMPETITIVE EQUILIBRIUM IN A DYNAMIC ECONOMY WITH NEGOTIABELE SHARES*

by

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Introduction.

In the economic model under consideration, productive activities of the firms take place during a sequence of periods with equal duration and in such a manner that inputs at the beginning of a period result in outputs which become available at the end of that period. Inputs are financed by individuals which also play the role of consumers. The rewards of the outputs of a firm at the end of a period are distributed among the individuals in the same proportion as each of them contributes in financing the inputs at the beginning of that period. Under the assumption that only firms are able to transfer goods from preceeding periods to succeeding periods, the exchange of goods and services between individuals and firms take place at the moments of period changing to be called time-points.

At each separate time-point, all agents are subject to budget constraints which are based on a price-system for each separate time-point. For an individual, the budget constraint requires that, at each time-point, the

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* This study was carried out at Cowles Foundation and was supported by the Netherlands Organization for the Advancement of Pure Research.

** The author is indebted to Herbert Scarf and Martin Shubik for helpful discussions.

value of his consumption and his contribution in financing inputs of firms may not exceed his income which is constituted by the value of his (labor) supply and his part of the rewards generated by the outputs of the preceeding period. For a firm, the budget constraint requires that, at each time-point, the value of its inputs may not exceed the budget generated by the contributions of individuals.

For each period, firms maximize the value of their outputs, by choosing a technologically feasible production which satisfies the budget constraints. This process also determines, for each firm and for each period, a dividend-factor defined as the ratio between output value and the end of a period and the budget at the start of that period.

Individuals maximize an ∞-horizon criterion function consisting of the discounted sum of single-period utility functions on consumption-supply combinations, by choosing a sequence of consumption, supply, and contributions in financing the firms. It will be shown that, in the context an "invariant competitive equilibrium", to be defined later, these ∞-horizon decision processes can be replaced by single-period decision processes.

It is assumed that the economic system is invariant over time, i.e.: the number of firms and individuals, the technology of the firms, individual's consumption-supply possibilities, and individual's preferences are the inchangeable over time. Then, the concept of invariant competitive equilibrium (briefly: I.C.E.) is defined as an invariant price-dividend system together with invariant action plans such that: (1) the action plans are compatible with the optimization behavior of the agents, (2) the dividend-factors represent that ratios between output-values and the budgets of the firms, (3) for each period, the total supply of commodities is equal to the total demand.

Under assumptions which correspond with Debreu's suppositions concerning production and consumption sets, and under concavity of the utility functions, the following results are deduced: (1) there exists an I.C.E., (2) the "physical" part of an I.C.E. is compatible with any degree of inflation or deflation, (3) under some additional assumptions, every I.C.E. is Pareto efficient.

The paper is organized as follows. Section 1 gives an axiomatic model of technology and of individual's consumption-supply possibilities, and analyzes the consequences of the overall balance of goods and services. In Section 2, the economic behavior of the firm is studied. In Section 3, we study the economic behavior of individuals. In addition, we show that, in invariant circumstances, the behavior can be characterized by a single-period decision process. Section 4 gives the definition of the concept I.C.E. and affirms the existence. Section 5 gives the results concerning inflation and Pareto efficiency. Section 6 contains the proofs. (Standard proofs are omitted).

The appendix (section 7.) contains some general properties concerning convex sets and sequences of difference inequalities. A list of symbols is added at the end.

The concept of I.C.E. might be considered as an hybrid of the usual competitive equilibrium concept and the concept of invariant optimal solutions in convex ∞-horizon programs (viz. [3], [5], [8]). The dynamic character of the I.C.E. and its simplicity looks to be very appropriate with respect to the study of other dynamic phenomena, like money and banking*

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* This aspect will be the topic of forthcoming studies by Shubik and Evers.

1. Basic Elements of the economic system.

1.1. Periods, time-points.

Economic activities take place at the sequence of "periods" with equal duration. The periods are numbered $t = 0,1,\ldots$

The period numbered 0 is considered as the last passed period. The moments of period changing are called "time-points". Time-points are indicated as "the start of period t", or as "the end of period t".

1.2. Commodities.

We assume that there is only a finite number g of distinguishable kinds of commodities. These specification is invariant over the periods. In addition, it is assumed that the quantity of any kind of commodity, at any time-point, can be any real non-negative number; implying that, at each time-point, the commodity-space can be represented by R^g_+. Quantities of goods will be considered at the time-points, only; they are denoted by non-negative vectors, endowed, sometimes, with a sub-index refering (dependent of the context) to the preceeding or the succeeding period.

1.3. Prices.

With each commodity k, real non-negative numbers $p_{k,t}$, $t = 1,2,\ldots$ are associated, representing prices at the starting points of the periods $t = 1,2,\ldots$. For all commodities together, the price-system at the start of a period t is represented by a vector $p_t \in R^g_+$. The value of a bundle of commodities $y \in R^g_+$, relative to a price-system p_t, is the inner product $p_t'y$.

The effective meaning of the concept "value" will be clarified in the context of the budget constraints of the agents.

1.4. Individuals: consumption and supply.

There are two kinds of agents: individuals and firms. With respect to the commodities, the activities of individuals consist of consumption and (labor) supply. We assume that there are m individuals, acting over all periods. It is assumed that the consumption-supply activities take place at the time-points. They are represented by sequences $\{(z_t^i, w_t^i)\}_{t=1}^{\infty} \subset R_+^{2\cdot g}$, $i = 1,2,\dots,m$, where (z_t^i, w_t^i) stands for the consumption and the supply, resp., of individual i at the start of period t. We assume that, for each time-point, individual's physical consumption-supply possibilities (i.e. appart from the total demand and supply of commodities) can be described by "consumption-supply sets" $C^i \subset R_+^{2\cdot g}$, $i = 1,2,\dots,m$, implying invariancy over time and independency with respect to other agents.

1.5. Assumption on the consumption-supply sets $\{C^i\}_{i=1}^m$.

1.5-A1 : $(0,0) \in C^i$ (possibility of inaction)

1.5-A2 : $(z^i, w^i) \in C^i \Rightarrow \forall\ \underline{w}^i \in R_+^g \mid \underline{w}^i \leqq w^i : (z^i, \underline{w}^i) \in C^i$ (free disposal).

1.5-A3 : $\exists\ \bar{w} \in R_+^g : \forall\ (z^i, w^i) \in C^i : w^i \leqq \bar{w}$ (boundedness of supply).

1.5-A4 : C^i is closed.

1.5-A5 : C^i is convex.

Later, the possibility of a particular consumption (non-satuation § 3.3) and of a particular supply (productive supply § 4.3) shall be added to these assumptions.

1.6. Firms: input, outputs, production sets.

Production is understood as an activity of transforming inputs at the start of a period into outputs which become available at the end of that period. In that context, firms are the only agents which choose and carry out productive activities. We assume that there is a fixed numer of n firms; the are indicated by an index $j = 1,2,\ldots,n$. The production plan of a firm j is denoted by a sequence $\{(x_t^j,y_t^j)\}_{t=1}^{\infty} \subset R_+^{2.g}$, where x_t^j represents the inputs at the start of period t and where y_t^j gives the outputs which become available at the end of t. We assume that, for each period, firm's technological input-output possibilities can be described by "production-sets" $F^j \subset R_+^{2.g}$, $j = 1,2,\ldots,n$, implying invariancy over time and independency with respect to activities of other agents.

1.7. Assumption on production sets $\{F^j\}_{j=1}^n$.

1.7-A1 : $(0,0) \in F^j$ (possibility of inaction).

1.7-A2 : $(x^j,y^j) \in F^j \Rightarrow \forall\ \underline{y}^j \in R_+^g \mid \underline{y}^j \leqq y^j : (x^j,\underline{y}^j) \in F^j$ (free disposal).

1.7-A3 : $(x^j,y^j) \in F^j \Rightarrow \forall\ \alpha > 1 : \exists\ \beta > 1 : (\alpha.x^j,\beta.y^j) \in F^j$ (possibility of expansion).

1.7-A4 : $(0,y^j) \in F^j \Rightarrow y^j = 0$ (no free production)

1.7-A5 : F^j is closed

1.7-A6 : F^j is convex.

<u>Proposition 1.7.1.:</u> The assumptions 1.7-A1, 4, 5 and 6, imply the existence of positive numbers α,β such that, for every $(x^j,y^j) \in F^j : \|y^j\| \leqq \alpha+\beta.\|x^j\|$.
(Direct consequence of auxiliary proposition 7.4.)

1.8. Total balance of goods; non-substitution of individual's supply.

Given the initial outputs $\{y_0^j\}_{j=1}^n$ all paths of actions $\{(\{(z_t^i,w_t^i)\}_{i=1}^m, \{(x_t^j,y_t^j)\}_{j=1}^n)\}_{t=1}^\infty$ have to satisfy:

$$(1.8.1) \quad \sum_{i=1}^m (z_t^i-w_t^i) + \sum_{j=1}^n (x_t^j-y_{t-1}^j) \leqq 0, \; t = 1,2,\ldots$$

Such a path will be called <u>feasible</u> (with respect to $\{y_0^j\}_{j=1}^n$) if, in addition: $\{(z_t^i,w_t^i)\}_{t=1}^\infty \subset C^i$, $i = 1,2,..m$ and $\{(x_t^j,y_t^j)\}_{t=1}^\infty \subset F^j$, $j = 1,2,\ldots n$.
Note: the time-lag with respect to the outputs appearing in (1.8.1) is caused by the fact that production takes exactly one period.

In the case of invariant actions $(z_t^i,w_t^i) := (z^i,w^i)$, $i = 1,2,\ldots,m$, $t = 1,2,\ldots$, and $(x_t^j,y_t^j) := (x^j,y^j)$, $j = 1,2,\ldots,n$, $t = 1,2,\ldots$ with, in addition, $y_0^j := y^j$, $j = 1,2,\ldots$, the balance of goods (1.8.1) reduces to:

$$(1.8.2) \quad \sum_{i=1}^m (z^i-w^i) + \sum_{j=1}^n (x^j-y^j) \leqq 0.$$

In that context we shall speak from an <u>feasible invariant state</u> if, in addition: $(z^i,w^i) \in C^i$, $i = 1,2,\ldots$ and $(x^j,y^j) \in F^j$, $j = 1,2,\ldots,n$.
For this case, it should be clear that the necessity of supply by individuals for invariant non-zero production is expressed by the following assumption.

1.8-A1: There is no invariant non-zero production $\{(x^j,y^j)\}_{j=1}^n$, $(x^j,y^j) \in F^j$, $j = 1,2,\ldots,n$, satisfying $\sum_{j=1}^n (x^j-y^j) \leqq 0$.

Note that 1.8-A1 implies the condition formulated by 1.7-A4.

Theorem 1.8.1.: Under the assumptions 1.5-A1, 3, 4, 5, 1.7-A1, 4, 5, 6, and assumption 1.8-A1, the set of feasible invariant states is bounded.

Theorem 1.8.2.: Under the assumptions 1.5-A1, 3, 4, 5, 1.7-A1, 2, 5, 6, and assumption 1.8-A1, for every initial state $\{y_0^j\}_{j=1}^n$ a number M exists such that all corresponding feasible paths $\{(\{(z_t^i,w_t^i)\}_{i=1}^m, \{(x_t^j,y_t^j)\}_{j=1}^n)\}_{t=1}^\infty$ satisfy: $\|y_t^j\| \leqq M$, $j = 1,2,\ldots,n$, $t = 1,2,\ldots$

Collary 1.8.3.: It should be clear that under the conditions, mentioned above, the consumption-supply sets C^i and the production sets F^j may be replaced by subsets $\bar{C}^i$, $\bar{F}^j$ defined by the relations:

1.8-D1: $\bar{C}^i := \{(z^i,w^i) \in C^i \mid z^i \leqq \bar{z}\}$, $i = 1,2,\ldots,m$,

1.8-D2: $\bar{F}^j := \{(x^j,y^j) \in F^j \mid x^j \leqq \bar{x}\}$, $j = 1,2,\ldots,n$,

provided the bounds $\bar{z},\bar{x}$ are taken large enough. Evidently we have the following properties:

Proposition 1.8.4.: Each of the assumptions 1.5-A1 to 4 implies an equivalent property with respect to the subsets $\bar{C}^i$. Moreover, 1.8-D1 and 1.5-A2 imply boundedness of each $\bar{C}^i$.

Proposition 1.8.5.: Each of the assumptions 1.7-A1, 2, 4, 5, and 6 implies an equivalent property with respect to the subsets $\bar{F}^j$. Moreover, 1.8-D2 and 1.7-A4 imply (by proposition 1.7.1) boundedness of each $\bar{F}^j$.

2. The economic behavior of the firms.

2.1. Budget constraints of the firm; shares.

The proprietary rights over a firm, during a period t, are distributed among individuals in the same proportion as each of them contributes in financing the inputs at the start of the period. These contributions, from now on to be called shares, will be represented by a sequence of non-negative m×n-matrices $\{S_t\}_{t=0}^{\infty}$; a matrix element $s_t^{i,j}$ stands for the shares owned by individual i, at the start of t, with respect to firm j. Thus, given a share distribution $\{S_t\}_{t=1}^{\infty}$ and a sequence of prices $\{p_t\}_{t=1}^{\infty}$, the budget constraints of the firms are formulated by: $p_t' \, x_t^j \leqq \sum_{i=1}^{m} s_t^{i,j}$, $j = 1,2,\ldots,n$, $t = 1,2,\ldots$ Further, we introduce the possibility that shareholding of some of the firms is open to only one or to part of the individuals. For instance stock holding activities of an individual might be described as production of a particular firm with exclusive proprietary rights. Assuming that these restrictions are invariant over time, we express the possibilities of share-holding by intervals $\omega^{i,j}$ such that $\omega^{i,j} := \{0\}$ in the case that the share-holding of firm j is closed for individual i, and $\omega^{i,j} := R_+^1$ otherwise. This aspect also will be expressed by a subset Ω in the space of real m×n-matrices $M^{m\times n}$, defined by: $\Omega := \{S \in M^{m\times n} \mid s^{i,j} \in \omega^{i,j},\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n\}$. Self-evident we assume that, for each firm, share-holding is open for at least one individual.

2.2. Choice criterion of the firm.

Given the prices $\{p_t\}_{t=1}^{\infty}$ and the shares $\{S_t\}_{t=1}^{\infty}$, the economic behavior of the firms is characterized by the programs:

2.2-D1.: $\sup p'_{t+1}y^j_t$, over $(x^j_t,y^j_t) \in F^j$,

$$\text{subject to: } p'_t x^j_t \leqq \sum_{i=1}^{m} s^{i,j}_t \qquad t = 1,2,\ldots$$

Provided that, for some $\sum_{i=1}^{m} s^{i,j}_t > 0$, $(\hat{x}^j_t,\hat{y}^j_t)$ is an optimal solution with respect a period t, the yields of the outputs over that period is distributed among share holders in the following way: $(s^{i,j}_t / \sum_{i=1}^{m} s^{i,j}_t).p'_{t+1}\,\hat{y}^j_t$, $i = 1,2,\ldots,m$.

Defining the concept of "dividend-factors" $\{\{d^j_t\}^n_{j=1}\}^{\infty}_{t=1}$ such that

2.2-D2: $$d^j_{t+1} \cdot \sum_{i=1}^{m} s^{i,j}_t = p'_{t+1}\,\hat{y}_t,\ j = 0,1,\ldots,n,\quad t = 1,2,\ldots$$

The payments of (liquidating) dividends to individual i can be written: $d^j_{t+1}.s^{i,j}_t$, $j = 1,2,\ldots,n$, $t = 0,1,2,\ldots$, where $d^j_1.s^{i,j}_0$ $j = 1,2,\ldots,n$ represents the payments resulting from productive activities of the initial period.

2.3. Economic behavior of the firm under invariant prices and shares.

Under invariant prices p and invariant shares S, the economic behavior of a firm j is described by 2.2-D1 with $(p_t,S_t) := (p,S)$, $t = 1,2,\ldots$ Limiting ourselves to bounded production sets $\{F^j\}^n_{j=1}$ (viz. 1.8-D2) and to an artificial maximum $\bar{\alpha}$ with respect to dividend-factors (to be clarified in § 4.4 and 4.2.4) we replace the max. program 2.2-D1 by:

2.3-D1: $\bar{\Psi}_j(p,S) := \sup p'y^j$, over $(x^j,y^j) \in \bar{F}^j$,

$$\text{subject to: } p'x^j \leqq \sum_{i=1}^{m} s^{i,j},\ p'y^j \leqq \bar{\alpha} \cdot \sum_{i=1}^{m} s^{i,j}.$$

The supremum $\bar{\Psi}_j$ will be taken as a function from $R^g_+ \times \Omega$ to R^1. Further, we introduce for each $j = 1,2,\dots,n$, a set-valued function $\hat{F}^j : R^g_+ \times \Omega \twoheadrightarrow R^{2\cdot g}_+$, defined by:

<u>2.3-D2:</u> $$\hat{F}^j(p,S) := \{(x^j,y^j) \in \bar{F}^j \mid p'x^j \leqq \sum_{i=1}^{m} s^{i,j},\ p'y^j = \bar{\Psi}_j(p,S)\}.$$

Clearly, for each $(p,S) \in R^g_+ \times \Omega$, the set $\hat{F}^j(p,S)$ gives the corresponding optimal solutions of 2.3-D1. The following properties can be deduced by standard methods:

<u>Propositions 2.3.1 to 2.3.4.:</u> The assumptions 1.7-A1, 2, 5, and 6 imply the following properties:

(2.3.1): $\forall (p,S) \in R^g_+ \times \Omega : \hat{F}^j(p,S) \neq \emptyset$, $j = 1,2,\dots,m$.

(2.3.2): The functions $\bar{\Psi}_j : R^g_+ \times \Omega \to R^1$ are concave and non-decreasing in the second argument.

(2.3.3): The functions $\bar{\Psi}_j$ are continuous in every $(p,S) \in R^g_+ \times \Omega$.

(2.3.4): The set-valued functions $\hat{F}^j : R^g_+ \times \Omega \twoheadrightarrow R^{2\cdot g}_+$ are upper-semi-continuous* and convex.

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* A set-valued function $\Gamma : X \twoheadrightarrow Y$ (X and Y normed spaces) is called upper-semi-continuous if, for any $\{(x^k,y^k)\}_0^\infty \subset X{\times}Y$, the relations $y^k \in \Gamma(x^k)$, $k = 1,2,\dots$ and $(x^k,y^k)\to(x^0,y^0)$ for $k\to\infty$, imply: $y^0 \in \Gamma(x^0)$. (see for instance Arrow and Hahn [1]).

Proposition 2.3.5.: Assumption 1.7-A3 implies the following property: If, for some $(p,S) \in R_+^g \times \Omega$: $\hat{F}^j(p,S) \neq \emptyset$, $\bar{\Psi}_j(p,S) > 0$, then every $(\hat{x}^j,\hat{y}^j) \in \hat{F}^j(p,S)$ satisfies at least one of the equalities: $p'\hat{x}^j = \Sigma_{i=1}^m s^{i,j}$ or $p'\hat{y}^j = \bar{\alpha}.\ \Sigma_{i=1}^m s^{i,j}$.

Finally, starting from the max. problems 2.3-D1, we introduce, with regard to the dividend-factors (viz. 2.2-D2), the set-valued functions $D^j : R_+^g \times \Omega \twoheadrightarrow R^1$ $(j = 1,2,\dots,n)$ by the relation:

2.3-D3.:

$$D^j(p,S) := \left\{ \delta \in [0,\bar{\alpha}] \;\middle|\; \begin{array}{l} \delta := \bar{\Psi}_j(p,S)/\Sigma_{i=1}^m s^{i,j}, \text{ in case } \Sigma_{i=1}^m s^{i,j} > 0, \\ \forall \underline{S} \in \Omega : \bar{\Psi}(p,\underline{S}) \leqq \delta.\Sigma_{i=1}^m \underline{s}^{ij}, \text{ otherwise} \end{array} \right\}$$

Clearly, this definition ensures that, for every $(p,S) \in R_+^g \times \Omega$ and every corresponding $d^j \in D^j(p,S)$, condition 2.2-D2 is satisfied.

Proposition 2.3.6.: If the assumptions 1.7-A1, 2, 5, and 6 are satisfied, then in every point $(p,S) \in R_+^g \times \Omega$ the set-valued functions D^j, $j = 1,2,\dots,n$ are non-empty, upper-semi-continuous and convex. In addition: $D^j(p,S) \in [0,\bar{\alpha}]$.

3. Economic Behavior of Individuals.

3.1. Introduction:

The budget constraints of individuals differ in one remarkable aspect from that of firms. Namely, individual's budget constraints on adjacent time-points are linked by the delay between payments and proceeds of shares. In fact, the entire dynamic character of the model is concentrated in this feature.

3.2. Budget constraints of individuals.

Given a sequence of prices $\{p_t\}_{t=1}^{\infty}$ and a sequence of dividend-factors $\{\{d_t^j\}_{j=1}^n\}_{t=1}^{\infty}$, the budget constraints of the individuals $i = 1,2,\ldots,m$ are formulated:

3.2-D1: $$p_t' z_t^i - p_t' w_t^i + \Sigma_{j=1}^n s_t^{i,j} - \Sigma_{j=1}^n d_t^j . s_{t-1}^{i,j} \leqq 0, \; t = 1,2,\ldots$$

The initial shares S_0 are supposed to be the given result of the initial periods $t := 0$.

Since the amounts of shares, an individual buys, constitute a part of his decision variables, the linked structure of the budget constraints, caused by share holding, implies that all budget constraints of an individual have to considered, simultaneously, over the whole time-horizon. However, in the case of invariant prices and invariant dividend-factors, we shall deduce that this system of budget constraints can be reduced to a particular single-period budget constraint.

3.3. Individual's choice criterion.

Within the sets of admissible actions (i.e. consumption, supply, and share holding) we assume that individual's choice criterions can be expressed by utility functions possessing the following structure*:

3.3-D1: $$\Sigma_{t=1}^h (\pi_i)^t . \varphi_i(z_t^i, w_t^i),$$

* For a fundamental study relating this structure to postulates about a preference ordering on a set of feasible actions, see Koopmans [6].

where the scalars $\pi_i \in]0,1[$ are time-discount factors, representing individual's time preference. The time-horizon h will be specified later. Assuming that prices and dividend-factors are the only information about the economic system as a whole, earning-capacity, being effectuated in the budget constraints, is the only attractive aspect of share-holding. For that reason, shares are not adopted in the utility function. Concerning the single-period utility function $\varphi_i : C^i \to R^1$, we make the following assumptions:

3.3-A1: φ_i is continuous.

3.3-A2: φ_i is concave

3.3-A3: $\varphi_i(0,0) = 0$

3.3-A4: $\exists\, \underline{z}^i \in R^g_+ : \forall (z^i,w^i) \in C^i,\ \lambda > 0 : (\lambda.\underline{z}^i+z^i,\ w^i) \in C^i$
$\varphi_i(\lambda.\underline{z}^i+z^i,w^i) > \varphi_i(z^i,w^i)$, (non-satuation condition);
a vector $\underline{z}^i$ with these properties will be called a non-satuation direction).

3.3-A5: $(z^i,\bar{w}^i) \in C^i,\ (z^i,\tilde{w}^i) \in C^i,\ \bar{w}^i \leqq \tilde{w}^i \Rightarrow$
$\varphi_i(z^i,\bar{w}^i) \geqq \varphi_i(z^i,w^i)$. (non-increasing w.r.t. supply).

A next aspect which has to be specified is the time-horizon h. Technically it makes sense to assume an infinite horizon, meeting the intuitive notion that individuals do not specify any terminal point, but, implying the dubious supposition that all individuals have, and actually use, full insight about future prices and dividends. Clearly, this objection can be relieved by assuming invariant prices and dividends. If, in addition, we restrict ourselves to invariant action plans under ∞-horizon optimization with invariant prices and dividends, then it turns out that the ∞-horizon decision processes can be reduced to single-period optimization problems.

Under general sequences of prices and dividend-factors, individual's ∞-horizon decision processes is characterized by:

3.3-D2: $\sup \Sigma_{t=1}^{\infty} (\pi_i)^t . \varphi_i(z_t^i, w_t^i)$, over $\{(z_t^i, w_t^i)\}_{t=1}^{\infty} \subset \bar{C}^i$, $\{(s_t^{i,1}, \ldots, s_t^{i,n})\}_{t=1}^{\infty} \subset (\omega^{i,1} \times \ldots \times \omega^{i,n})$, subject to:

$$p_t' z_t^i - p_t' w_t^i + \Sigma_{j=1}^{n} (s_t^{i,j} - d_t^j . s_{t-1}^{i,j}) \geqq 0, \; t = 1,2,\ldots$$

<u>3.4. Infinite horizon decision processes versus single-period optimization.</u>

Given an invariant price-dividend system $(p, \{d^j\}_{j=1}^{n})$ and given an initial share distribution S_0, we characterize individual's economic behavior by:

3.4-D1: $\sup \Sigma_{t=1}^{\infty} (\pi_i)^t . \varphi_i(z_t^i, w_t^i)$, over $\{(z_t^i, w_t^i)\}_{t=1}^{\infty} \subset \bar{C}^i$, $\{(s_t^{i,1}, s_t^{i,2}, \ldots, s_t^{i,n})\}_{t=1}^{\infty} \subset (\bar{\omega}^{i,1} \times \bar{\omega}^{i,2} \times \ldots \times \bar{\omega}^{i,n})$,

subject to: $p' z_t^i - p' w_t^i + \Sigma_{j=1}^{n} (s_t^{i,j} - d^j . s_{t-1}^{i,j}) \leqq 0, \; t = 1,2,\ldots$

where $\bar{C}^i$ are the bounded consumption-supply sets (viz. 1.8.3) and where $\bar{\omega}^{i,j} := \omega^{i,j} \cap [0, \bar{\omega}]$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$, $\bar{\omega}$ being some artificial upperbound on share holding to be clarified in § 4.4.

We define the corresponding single-period optimization problems as follows:

<u>3.4-D2:</u> $\mu_i(p, d^1, d^2, \ldots, d^n, S_0) := \sup \varphi_i(z^i, w^i)$, over $(z^i, w^i) \in \bar{C}^i$, $(s^{i,1}, s^{i,2}, \ldots, s^{i,n}) \in (\bar{\omega}^{i,1} \times \bar{\omega}^{i,2} \times \ldots \times \bar{\omega}^{i,n})$, s.t.:

$$p' z^i - p' w^i + \Sigma_{j=1}^{n} (1 - \pi_i . d^j) . s^{i,j} \leqq (1 - \pi_i) . \Sigma_{j=1}^{n} d^j . s_0^{i,j}.$$

<u>Propositions 3.4.1 to 3.4.4.:</u> Under 1.5-A3: For every admissible action $\{((z_t^i, w_t^i), \{s_t^{i,j}\}_{j=1}^{n})\}_{t=1}^{\infty}$ of 3.4-D1, the following properties hold:

(3.4.1): The series $\{\Sigma_{t=1}^{h} (\pi_i)^t.(z_t^i, w_t^i, s_t^{i,1}, \ldots, s_t^{i,n})\}_{h=1}^{\infty}$ converges; in the next propositions we denote the limit point: $(\tilde{z}^i, \tilde{w}^i, \{\tilde{s}^{i,j}\}_{j=1}^{n})$.

(3.4.2): Under 1.5-A4 and 1.5-A5: $((1-\pi_i)/\pi_i).(\tilde{z}^i, \tilde{w}^i, \{\tilde{s}^{i,j}\}_{j=1}^{n})$ is an admissible action w.r.t. 3.4-D2, provided the initial share distribution is the same.

(3.4.3): Under 1.5-A4, and 3.3-A1: the series $\{\Sigma_{t=1}^{h} (\pi_i)^t.\varphi_i(z_t^i, w_t^i)\}_{h=1}^{\infty}$ converges.

(3.4.4): Defining $\lambda_i := (1-\pi_i)/\pi_i$, the assumptions 1.5-A4, 3.3-A1, and 3.3-A2 imply: $\varphi_i(\lambda_i.\tilde{z}^i, \lambda_i.\tilde{w}^i) \geqq \lambda_i.\Sigma_{t=1}^{\infty} (\pi_i)^t.\varphi_i(z_t^i, w_t^i)$.

<u>Theorem 3.4.5:</u> Under 1.5-A3, 1.5-A4, 1.5-A5, 3.3-A1, and 3.3-A2, we have the following property: If, for any initial share distribution S_0, $(\hat{z}^i, \hat{w}^i\{s^{i,j}\}_{j=1}^{n})$ is an optimal solution of the single-period program 3.4-D2 which satisfies $\hat{s}^{i,j} = s_0^{i,j}$, $j = 1,2,\ldots,n$, then, for the same initial shares $\{s_0^{i,j}\}_{j=1}^{n}$, $(\hat{z}_t^i, \hat{w}_t^i, \hat{s}_t^{i,1}, \ldots, \hat{s}_t^{i,n}) := (\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n})$, $t = 1,2,\ldots$ constitutes an optimal solution w.r.t. the ∞-horizon program 3.4-D1.

Note: the opposite is <u>not</u> stated; i.e. an invariant optimal action plan w.r.t. 3.4-D1 will not necessarily generate an optimal solution w.r.t. 3.4-D2. Anyway, theorem 3.4.5 ensures that the "best" invariant ∞-horizon actions will be selected by the single-period decision processes with the appropriate initial share distributions. For that reason we accept the single-period programs as adequate descriptions of the economic behavior of the individuals under invariant prices and dividends.*

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* For a fundamental study concerning convex ∞-horizon programming, see Evers [4].

The optimal solutions of 3.4-D2 will be represented by the set-valued functions, $B^i : R^g_+ \times R^n_+ \times \Omega \twoheadrightarrow R^{2.g}_+ \times \bar{\omega}^{i,1} \times \ldots \times \bar{\omega}^{i,n}$, defined by:

3.4-D3:
$$B^i(p,d^1,\ldots,d^n,S_0) := \left\{ (z^i,w^i,s^{i,1},\ldots,s^{i,n}) \in \bar{C}^i \times \bar{\omega}^{i,1} \times \ldots \times \bar{\omega}^{i,n} \middle| \begin{array}{l} \varphi_i(z^i,w^i) = \mu_i(p,d^1,\ldots,d^n,S_0), \\ p'z^i - p'w^i + \Sigma^n_{j=1}(1-\pi_i.d^j).s^{i,j} \leqq (1-\pi_i).\Sigma^n_{j=1} d^j.s^{i,j}_0 \end{array} \right\}$$

Using standard methods, one can deduce the following properties:

Proposition 3.4.6.: Under the assumptions 1.5-A1, 3, 4, 5 and the assumptions 3.3-A1, 2: At every $(p,d^1,\ldots,d^n,S_0) \in R^g_+ \times R^n_+ \times \Omega$ the functions B^i are non-empty and convex. If, in addition, the point $(p,d^1,\ldots,d^n,S_0)$ is such that a $(z^i,w^i) \in \bar{C}^i$ exists satisfying $p'z^i - p'w^i < 0$, then B^i also is upper-semi-continuous in that point of the domain.

4. Invariant Competitive Equilibrium. (I.C.E).

4.1. Definition.

We define "invariant competitive equilibrium" (abbreviated: I.C.E.) as a combination of (1) prices $\hat{p} \in R^g_+$, (2) dividend factors $\{\hat{d}^j\}^n_{j=1}$, (3) shares $\hat{S} \in \Omega$, (4) consumption-supply $(\hat{z}^i,\hat{w}^i) \in C^i$, $i = 1,2,\ldots,m$, and (5) inputs-outputs $(\hat{x}^j,\hat{y}^j) \in F^j$, $j = 1,2,\ldots,n$, such that simultaneously:

a) Each $(\hat{z}^i,\hat{w}^i,\{\hat{s}^{i,j}\}^n_{j=1})$ is optimal with respect to:
   $\sup \varphi_i(z^i,w^i)$, over $(z^i,w^i) \in C^i$, $s^{i,j} \in \omega^{i,j}$, $j = 1,2,\ldots,n$,
   subject to: $\hat{p}'z^i - \hat{p}'w^i + \Sigma^n_{j=1}(1-\pi_i.d^j).s^{i,j} \leqq (1-\pi_i).\ \Sigma^n_{j=1}\ d^j.\hat{s}^{i,j}$.
   (As pointed out in §3.4, such an optimal solution may be considered as an invariant optimal solution of the ∞-horizon process 3.4-D1)

(b) Each $(\hat{x}^j,\hat{y}^j)$ is optimal with respect to:

$\sup \hat{p}'y^j$, over $(x^j,y^j) \in F^j$, subject to: $\hat{p}'x^j \leqq \Sigma_{i=1}^m \hat{s}^{i,j}$.

(c) The dividend-factors $\{\hat{d}^j\}_{j=1}^n$ satisfy:

$\hat{p}'\hat{y}^j = \hat{d}^j.\Sigma_{i=1}^m \hat{s}^{i,j}$, $j = 1,2,\ldots,n$.

(d) Total demand and total supply of commodities are equal; i.e.:

$\Sigma_{i=1}^m (\hat{z}^i-\hat{w}^i) + \Sigma_{j=1}^n (\hat{x}^j-\hat{y}^j) = 0$.

Under certain assumptions (viz. 4.2.8), condition (d) can be replaced by:

(d'). Total demand is not larger than total supply of commodities; i.e.

$\Sigma_{i=1}^m (\hat{z}^i-\hat{w}^i) + \Sigma_{j=1}^n (\hat{x}^j-\hat{y}^j) \leqq 0$.

A direct consequence of the equilibrium conditions is the following homogeneity property:

<u>Proposition 4.1.1.:</u> If $(\hat{p},\{\hat{d}^j\}_1^n, \hat{S},\{(\hat{z}^i,\hat{w}^i)\}_1^m, \{(\hat{x}^j,\hat{y}^j)\}_1^n)$ is an I.C.E. then for every $\lambda > 0$, $(\lambda.\hat{p},\{\hat{d}^j\}_1^n, \lambda.\hat{S}, \{(\hat{z}^i,\hat{w}^i)\}_1^m, \{(\hat{x}^j,\hat{y}^j)\}_1^n)$ is an I.C.E., as well.

Previous to prove the existence of I.C.E., we deduce a couple of auxillary properties which also contain some economic relevance.

<u>4.2. Non-satuation, consistency of the price-dividend system, and Walras' law.</u>

In order to introduce some necessary conditions for invariant price-dividend systems to appear in an I.C.E., we call a price-dividend system $(p,\{d^j\}_1^n)$ <u>consistent</u> if, simultaneously:

(a) For all directions of non-satuation $\{\underline{z}^i\}_1^m$ (viz. 3.3-A4): $p'\underline{z}^i > 0$.

(b) For all $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ with $\omega^{i,j} \neq \{0\}$: $\pi_i.d^j \leqq 1$.

(c) A number $\underline{\omega} > 0$ exists such that, for each $j = 1,2,\dots,n$, the relations $(x^j,y^j) \in F^j$, $p'x^j \leqq \underline{\omega}$ imply $p'y^j \leqq d^j.\underline{\omega}$.

Obvious, removing the artifical bounds in individual's decision process (3.4-D1) or (3.4-D2), violating (a) and/or (b), under non-satuation (3.3-A4), implies that at least one individual would be able to increase his utility by increasing his consumption above any bound. Formally, this would imply that the total balance of goods expressed by equilibrium condition (d) or (d') cannot be satisfied. Violating (c) means that, under price system p, there is at least one firm j with a profit ration higher than d^j; implying that d^j cannot be a well-defined dividend-factor. Thus, we have:

Proposition 4.2.1.: Under 3.3-A4 (i.e. non-satuation), consistency is a necessary condition for an invariant price-dividend system to be a part of an I.C.E.

Consistency condition (b) gives rise to scalars $\{\rho_j\}_1^n$ defined by:

4.2-D1: $\rho_j := \min(1/\pi_i)$, over $i = 1,2,\dots,m$, s.t.: $\omega^{i,j} \neq \{0\}$.

Now we can formulate the following consequences of the definitions:

Proposition 4.2.2. to 4.2.5.: Under 3.3-A4, consistency of an invariant price-dividend system implies:

(4.2.2): $p \neq 0$

(4.2.3): Taking the artificial bounds in 3.4-D2 large enough, necessary conditions for feasible solutions $(z^i,w^i,\{s^{i,j}\}_{j=1}^n)$ of 3.4-D2 to be optimal are:

$$p'z^i - p'w^i + \Sigma_{j=1}^n\, (1-\pi_i.d^j).s^{i,j} = (1-\pi_i).\Sigma_{j=1}^n\, s_0^{i,j},$$

$$s^{i,j} = 0 \text{ in case } \pi_i.d^j < 1.$$

(4.2.4): $d^j \leqq \rho^j$, $j = 1,2,\ldots,n$

(4.2.5): If an $(\underset{\sim}{x}^j,\underset{\sim}{y}^j) \in F^j$ exists such that, for all $\lambda \geqq 0$, $\cdot(\underset{\sim}{x}^j,\underset{\sim}{y}^j) \in F^j$ then $p'\underset{\sim}{y}^j \leqq d^j.p'\underset{\sim}{x}^j$.

With the help of these properties one can find:

Theorem 4.2.6.: Under 1.7-A3 and 3.3-A4: Every combination $(\hat{p},\{\hat{d}^j\}_1^n, \hat{S},\{(\hat{z}^i,\hat{w}^i)\}_1^m,\{(\hat{x}^j,\hat{y}^j)\}_1^n)$ which satisfies the equilibrium conditions (a), (b), (c), (d'), also satisfies: $\hat{p}'\hat{x}^j = \Sigma_{i=1}^m \hat{s}^{i,j}$.
(i.e.: firms fully use their budgets).

Theorem 4.2.7.: Under 1.7-A3 and 3.3-A4: Every combination $(\hat{p},\{\hat{d}^j\}_1^n, \hat{S},\{(\hat{z}^i,\hat{w}^i)\}_1^m,\{(\hat{x}^j,\hat{y}^j)\}_1^n)$, $(\hat{p},\{\hat{d}^j\}_1^n)$ being consistent, which satisfies the equilibrium conditions (a), (b), (c), also satisfies:
$\hat{p}'(\Sigma_{i=1}^m (\hat{z}^i-\hat{w}^i) + \Sigma_{j=1}^n (\hat{x}^j-\hat{y}^j)) = 0$. (Analogy of Walras' law).

Theorem 4.2.8.: Under 1.5-A2, 1.7-A2, 1.7-A3, 3.3-A4, and 3.3-A5: For every $(\hat{p},\{\hat{d}^j\}_1^n, \hat{S},\{(\hat{z}^i,\hat{w}^i)\}_1^m,\{(\hat{x}^j,\hat{y}^j)\}_1^n)$ which satisfies the equilibrium conditions (a), (b), (c), (d'), vectors $\bar{w}^i \in [0,\hat{w}^i]$, $i = 1,2,\ldots,m$ and $\bar{y}^j \in [0,\hat{y}^j]$, $j = 1,2,\ldots,n$ exist such that
$(\hat{p},\{\hat{d}^j\}_1^n, \hat{S},\{(\hat{z}^i,\bar{w}^i)\}_1^m,\{(\hat{x}^j,\bar{y}^j)\}_1^n)$ is an I.C.E. (i.e. satisfies the equilibrium conditions (a) to (d).

4.3. Individual's productive supply capacity.

Although 4.2.2. ensures that, under non-satuation, the price system is non-zero, the possibility of an I.C.E. where none of the consumers earns any income is still present. Next assumption, to be called "productive supply capacity" rules out the possibility of such an equilibrium and, at the same time, indicates how an equilibrium with complete inaction can be excluded.

4.3-A1: For each i, a supply $\underset{\sim}{w}^i \neq 0$, $(0,\underset{\sim}{w}^i) \in C^i$ exists such that there is a $\{(\underset{\sim}{x}^j,\underset{\sim}{y}^j)\}_1^n$ satisfying the conditions:

(a) $\forall\ \lambda \geqq 0$: $\lambda.(\underset{\sim}{x}^i,\underset{\sim}{y}^i) \in F^j$, $j = 1,2,\ldots,n$

(b) $\Sigma_{j=1}^n(\underset{\sim}{y}^j-\rho_j\underset{\sim}{x}^j) + \underset{\sim}{w}^j > 0$, $\{\rho_j\}_1^n$ being the scalars defined by 4.2-D1.

Clearly, condition (a) states that $\{(\underset{\sim}{x}^j,\underset{\sim}{y}^j)\}_1^n$ can be produced in any multiple; (b) requires a net output of all kinds of commodities with a rate of productivity which is high enough to attract share holders.

Theorem 4.3.1.: If the assumptions 3.3-A4 and 4.3-A1 are satisfied, then a number $\nu > 0$ exists such that, for every consistent price-dividend system $(p,\{d^j\}_{j=1}^n)$, there are vectors $\{\underset{\sim}{w}^i\}_1^m$ satisfying: $(0,\underset{\sim}{w}^i) \in C^i$, $p'\underset{\sim}{w}^i \geqq \nu.\|p\|_1$, $i = 1,2,\ldots,m$.

Theorem 4.3.2.: If the assumptions 1.5-A1, 1.5-A5, and 3.3-A1 to 4 are satisfied, and if, for some individual i, there is a supply $\underset{\sim}{w}^i$ as described in 4.3-A1 such that $\varphi_i(0,\lambda.\underset{\sim}{w}^i)/\lambda \to 0$ for $\lambda \to^+ 0$ then, under a consistent price-dividend system $(\hat{p},\{\hat{d}^j\}_1^n)$, a necessary condition for $(\hat{z}^i,\hat{w}^i,\{\hat{s}^{i,j}\}_{j=1}^n)$ to satisfy equilibrium condition (a) is: $\hat{z}^i \neq 0$, $\hat{w}^i \neq 0$.

4.4. The existence of an invariant competitive equilibrium.

We start with some auxiliary definitions and proposition. First of all, we observe that, by virtue of 4.2.1., 4.1.1., and 4.2.2., we may restrict ourselves to invariant prices in the set:

<u>4.4-D1:</u> $P := \{p \in R^g_+ \mid \|p\|_1 = 1\}$.

Earlier we deduced that, under a number of assumptions, we may limit ourselves to the bounded subsets $\{\bar{C}^i\}^m_1, \{\bar{F}^j\}^n_1$ (viz. 1.8.3) and to dividend-factors in an interval $[0, \bar{\alpha}]$ (viz. 2.3-D1 and 4.2.4).
With bounded production sets $\{\bar{F}^j\}^n_1$ and with prices $p \in P$, theorem 4.2.6. implies that only shares $\{\{s^{i,j}\}^m_{i=1}\}^n_{j=1}$ in bounded intervals $\{\{\bar{\omega}^{i,j}\}^m_{i=1}\}^n_{j=1}$ have to be considered (viz. 3.4-D1, provided $\bar{\omega}$ is chosen large enough). Summarizing these arguments we have:

<u>Proposition 4.4.1.:</u> Under 1.5-A3, 1.7-A1 to 6, 1.8-A4, and 3.3-A4: a combination $(\hat{p}, \{\hat{d}^j\}^n_1, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}^m_1, \{(\hat{x}^j, \hat{y}^j)\}^n_1)$, with $\hat{p} \in P$, satisfies the equilibrium conditions (a), (b), (c), (d') if and only if, simultaneously:

(1) $(\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n}) \in B^i(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S})$, $i = 1,2,\ldots,m$.

(2) $(\hat{x}^j, \hat{y}^j) \in \hat{F}^j(\hat{p}, \hat{S})$, $j = 1,2,\ldots,n$.

(3) $\hat{d}^j \in D^j(\hat{p}, \hat{S})$, $j = 1,2,\ldots,n$.

(4) $\Sigma^m_{i=1}(\hat{z}^i - \hat{w}^i) + \Sigma^n_{j=1}(\hat{x}^j - \hat{y}^j) \leqq 0$.

In order to preserve the upper-semi continuity property of the set-valued functions B^i (viz. 3.4-D3) describing individual's economic behavior, we introduce set-valued functions $\bar{B}^i$, operating on the same domain, by the relation:

<u>4.4-D2:</u>

$$\bar{B}^i(p, d^1, \ldots, d^n, S_0) :=$$
$$:= \Big\{ (z^i, w^i, s^{i,1}, \ldots, s^{i,n}) \in \bar{C}^i \times \bar{\omega}^{i,1} \times \ldots \times \bar{\omega}^{i,n} \mid$$
$$(z^i, w^i, s^{i,1}, \ldots, s^{i,n}, p, d^1, \ldots, d^n, S_0) \in$$
$$\in \text{closure } (\text{graph } (B^i;\ R^g_+ \times R^n_+ \times \Omega)) \Big\}.$$

Adding the assumption 4.3-A1 to the assumptions of proposition 4.4.1., theorem 4.3.1. and proposition 3.4.6. imply that condition (1) of proposition 4.4.1. can be replaced by:

(1') $(\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \ldots, \hat{s}^{i,n}) \in \bar{B}^i(\hat{p}, \hat{d}^1, \ldots, \hat{d}^n, \hat{S})$.

Combining the set-valued functions $\bar{B}^i$, $\hat{F}^j$, and D^j, we define $G : P \times [0,\bar{\alpha}]^n \times \bar{\Omega} \twoheadrightarrow [0,\bar{\alpha}]^n \times \bar{\Omega} \times V$ by the relations:

<u>4.4-D3:</u> $G(\underline{p}, \underline{d}^1, \ldots, \underline{d}^n, \underline{S}) :=$

$$:= \Big\{(d^1, \ldots, d^n, S, v) \in [1,\bar{\alpha}]^n \times \bar{\Omega} \times R^g \;\Big|\; \exists\{(z^i,w^i)\}_1^m \subset R^{2\cdot g}, \{(x^j,y^j)\}_1^n \subset R^{2\cdot g}:$$
$$(z^i, w^i, s^{i,1}, \ldots, s^{i,n}) \in \bar{B}^i(\underline{p}, \underline{d}^1, \ldots, \underline{d}^n, \underline{S}),\ i = 1,2,\ldots,n$$
$$(x^j, y^j) \in \hat{F}^j(\underline{p}, \underline{S}),\ d^j \in D^j(\underline{p}, \underline{S}),\ j = 1,2,\ldots,n,$$
$$v = \Sigma_{i=1}^m (z^i - w^i) + \Sigma_{j=1}^n (x^i - y^i) \Big\}.$$

The set V, appearing in the domain is defined by:

<u>4.4-D4:</u> $V := \{v \in R^g \mid \|v\|_1 \leqq \beta\}$,

where β is such large that, for every $(z,w) \in \Sigma_{i=1}^m \bar{C}^i$, $(x,y) \in \Sigma_{j=1}^n \bar{F}^j$, $(z + w + x + y) \in V$.

Clearly, by 2.3.1., 2.3.4., 2.3.6., 3.4.6., 4.4-D2, and 4.4-D3 we may conclude that, under the appropriate assumptions, the set-valued function G is non-empty, convex, and upper-semi continuous in every point of its domain.

Finally we introduce the set-valued functions $\hat{P} : V \twoheadrightarrow P$ and $H : P \times [0,\bar{\alpha}]^n \times \bar{\Omega} \times V \twoheadrightarrow P \times [0,\bar{\alpha}]^n \times \bar{\Omega} \times V$ by the relations:

<u>4.4-D5:</u> $\hat{P}(\underline{v}) := \{\hat{p} \in P \mid \hat{p}'\underline{v} = (\max p'\underline{v}, \text{ over } p \in P)\}$

<u>4.4-D6:</u> $H(\underline{p},\underline{d}^1,\ldots,\underline{d}^n, \underline{S}, \underline{v}) :=$

$$:= \{(p.d^1,\ldots,d^n, S,v) \in \hat{P}(\underline{v}) \times G(\underline{p},\underline{d}^1,\ldots,\underline{d}^n, \underline{S})\}.$$

Since $\hat{P} : V \twoheadrightarrow P$ is non-empty, convex and upper-semi continuous in every point of its domain, and since the same can be said from the set-valued function G, we have:

<u>Proposition 4.4.2:</u> Under the assumptions 1.5-A1 to 5, 1.7-A1, 2, 5, 6 and 3.3-A1, 2: The set-valued function H (viz. 4.4-D6) is non-empty, convex, and upper-semi continuous in every point of its domain.

<u>Proposition 4.4.3 and 4.4.4:</u> Under the assumptions 1.5-A1 to 5, 1.7-A1 to 6, 1.8-A1, 3.3-A1 to 5, and 4.3-A1:

(4.4.3) A combination $(\hat{p},\{\hat{d}^j\}_1^n, \hat{S},\{(\hat{z}^i,\hat{w}^i)\}_1^m)$, with $p \in P$, satisfies the equilibrium conditions (a), (b), (c), (d') if, and only if, $(\hat{p},\{\hat{d}^j\}_1^n, \hat{S},\hat{v})$, with $\hat{v} := \Sigma_{i=1}^m(\hat{z}^i-\hat{w}^i) + \Sigma_{j=1}^n(\hat{x}^j-\hat{y}^j)$, is a fixed-point of H; i.e. $(\hat{p},\hat{d}^1,\ldots,\hat{d}^n,\hat{S},\hat{v}) \in H(\hat{p},\hat{d}^1,\ldots,\hat{d}^n, \hat{S},\hat{v})$.

(4.4.4) The set-valued function H possesses a fixed-point.

Using standard methods (see for instance Debreu [2]), proposition (4.4.3) can be found by elaboration of the definitions 4.4-D2 to 6 and proposition (4.4.1). By virtue of Kakutani's fixed-point theorem, proposition (4.4.2) follows from (4.4.2) and the fact that the domain of H is non-empty, convex, and compact.

A direct consequence of (4.4.3), (4.4.4), and (4.2.8) is the following existence theorem:

Theorem 4.4.5: Under the assumptions 1.5-A1 to 5, 1.7-A1 to 6, 1.8-A1, 3.3-A1 to 5, and 4.3-A1, there exists an I.C.E.

5. Particular properties of an I.C.E.: inflation and deflation, Pareto efficiency.

5.1. Inflation and deflation.*

Considering the definition of the decision prosesses 2.2-D1 and 3.3-D2, one can deduce the following properties:

Proposition 5.1.1: Let $\{(\tilde{p}_t,\{\tilde{d}_t^j\}_{j=1}^n)\}_{t=1}^{\infty}$ be sequence of prices and dividend-factors. Let $\{(z_t^i,w_t^i,\{s_t^{i,j}\}_{j=1}^n)\}_{t=1}^{\infty}$ be an action plan which is admissible (or optimal) w.r.t. 3.3-D2 under the initial shares $\{s_0^{i,j}\}_{j=1}^n$ and the price-dividend system $(p_t,d_t^1,\ldots,d_t^n) := (\tilde{p}_t,\tilde{d}_t^1,\ldots,\tilde{d}_t^n)$, t = 1,2,....
Then, for every sequence of positive numbers $\{\gamma_t\}_{t=0}^{\infty}$ with $\gamma_0 := 1$, the action plan $\{(z_t^i,w_t^i,\gamma_t.s_t^{i,1},\ldots,\gamma_t.s_t^{i,n})\}_{t=1}^{\infty}$ is admissible (or optimal) w.r.t. 3.3-D2 under the same initial shares and the price-dividend system $(p_t,d_t^1,\ldots,d_t^n) := (\gamma_t.\tilde{p}_t,\ (\gamma_t/\gamma_{t-1}).\tilde{d}_t^1,\ldots,(\gamma_t/\gamma_{t-1}).\tilde{d}_t^n)$, t = 1,2,...

Proposition 5.1.2.: Let $\{(\tilde{p}_t,\tilde{S}_t)\}_{t=1}^{\infty}$ be a sequence of prices and shares. Let $\{(x_t^j,y_t^j)\}_{t=1}^{\infty}$ be an action plan which is admissible (or optimal) w.r.t.

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* This section is based on a suggestion of Shubik.

2.2-D1 with $(p_t, s_t^{i,j}, \dots, s_t^{m,j}) := (\tilde{p}_t, \tilde{s}_t^{1,j}, \dots, \tilde{s}_t^{m,j})$, $t = 1,2,\dots$.
Then, for every sequence of positive number $\{\gamma_t\}_1^\infty$, the same action remains admissible (or optimal) under $(p_t, s_t^{i,j}, \dots, s_t^{m,j}) := \gamma_t.(\tilde{p}_t, \tilde{s}_t^{1,j}, \dots, \tilde{s}_t^{m,j})$, $t = 1,2,\dots$. In addition, concerning the dividend-factors the relations $\tilde{p}'_{t+1}\, y_t^j = \tilde{d}_{t+1}^j \cdot \Sigma_{i=1}^m\, \tilde{s}_t^{i,j}$, $t = 0,1,\dots$ imply:
$\gamma_{t+1}.\tilde{p}'_{t+1}\, y_t^j = (\gamma_{t+1}/\gamma_t).\tilde{d}_{t+1}^j.\Sigma_{i=1}^m\, \gamma_t.\tilde{s}_t^{i,j}$, $t = 0,1,\dots$

Now, starting from an I.C.E. $(\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{z}^j, \hat{y}^j)\}_1^n)$ and describing inflation (or deflation) with the help of positive numbers $\{\gamma_t\}_0^\infty$, $\gamma_0 := 1$ in the following manner:

<u>5.1-D1:</u> $\qquad p_t := \gamma_t.\hat{p}$, $t = 1,2,\dots,$

the fact that an I.C.E. generates invariant optimal action plans gives rise to the following property:

<u>Theorem 5.1.3:</u> Let $(\hat{p}, \{\hat{d}^j\}_1^n, \hat{S}, \{(\hat{z}^i, \hat{w}^i)\}_1^m, \{(\hat{x}^j, \hat{y}^j)\}_1^n)$ be an I.C.E.
Then, for every sequence of positive numbers $\{\gamma_t\}_0^\infty$, $\gamma_0 := 1$:

(1) for each $i = 1,2,\dots,m$, the action plans $(z_t^i, w_t^i, s_t^{i,1}, \dots, s_t^{i,n}) := (\hat{z}^i, \hat{w}^i, \gamma_t.\hat{s}^{i,1}, \dots, \gamma_t.\hat{s}^{i,n})$, $t = 1,2,\dots$ are optimal w.r.t. 3.3-D2 with $(s_0^{i,1}, \dots, s_0^{i,n}) := (\hat{s}^{i,1}, \dots, \hat{s}^{i,n})$.
$p_t := \gamma_t.\hat{p}$, $t = 1,2,\dots,$ and $d_t^j := (\gamma_t/\gamma_{t-1}).\hat{d}^j$, $j = 1,2,\dots,n$, $t = 1,2,\dots$

(2) for each $j = 1,2,\dots,m$, the action plans $(x_t^j, y_t^j) := (\hat{x}^j, \hat{y}^j)$, $t = 1,2,\dots$ are optimal for 2.2-D1 with $p_t := \gamma_t.\hat{p}$, $t = 1,2,\dots$ and $s_t^{i,j} := \gamma_t.\hat{s}^{i,j}$, $i = 1,2,\dots,m$, $t = 1,2,\dots$

(3) $\gamma_{t+1}.\hat{p} = (\gamma_t/\gamma_{t-1}).\hat{d}^j.\ \Sigma_{i=1}^m\ \hat{s}^{i,j}$, $j = 1,2,\ldots,n$, $t = 1,2,\ldots$

Briefly, the "physical" part of an I.C.E. is compatible with any degree of inflation or deflation.

5.2. Pareto efficiency.

In order to study Pareto efficiency in the context of ∞-horizon action plans, we introduce two optimality criteria: Given the initial outputs $\{y_0^j\}_{j=0}^n$, a feasible path $\{(\{(z_t^i,\ w_t^i)\}_{i=1}^m,\ \{(x_t^j,y_t^j)\}_{j=1}^n)\}_{t=1}^\infty$ (viz. § 1.8) is called strictly efficient if no feasible path $\{(\{(\bar{z}_t^i,\bar{w}_t^i)\}_{i=1}^m,\{(\bar{x}_t^j,\bar{y}_t^j)\}_{j=1}^n)\}_{t=1}^\infty$ exists such that:

(a) $\Sigma_{t=1}^\infty\ (\pi_i)^t.\varphi_i(\bar{z}_t^i,\bar{w}_t^i) \geqq \Sigma_{t=1}^\infty\ (\pi_i)^t.\varphi_i(z_t^i,w_t^i)$, $i = 1,2,\ldots,m$,
with strict inequality for at least one i.

In the concept of weak efficiency these condition is replaced by:

(b) $\varphi_i(\bar{z}_t^i,\bar{w}_t^i) \geqq \varphi_i(z_t^i,w_t^i)$, $i = 1,2,\ldots,m$, $t = 1,2,\ldots$
with strict inequality for at least one pair (i,t).

Note that feasible paths are bounded (viz. 1.8.2), implying that the ∞-horizon utility functions in (a) are well-defined. Clearly, strict efficiency implies weak efficiency.

In the next part it will be shown that, under some extra assumptions, every path generated by an I.C.E. is weakly or strictly efficient. These assumptions are:

5.2-A1: For each F^j: $\lambda \geqq 0$, $(x^j,y^j) \in F^j$ implies: $\lambda.(x^j,y^j) \in F^j$.
(i.e. linearity of the technology).

5.2-A2: The numbers $\{\rho_j\}_{j=1}^{n}$, defined by 4.2-D1, are equal.

5.2-A3: The time-discount factors $\{\pi_i\}_{i=1}^{m}$ are equal.

(Note: 5.2-A3 implies 5.2-A2).

The following auxiliary propositions can be deduced easily:

Proposition 5.2.1.: Consider, for each $j = 1,2,\dots,n$, the programs:

(a) $\sup p'y^j$, over $(x^j,y^j) \in F^j$, s.t. $p'x^j \leqq \beta$.

(b) $\inf(-p'y^j + \delta.p'x^j)$, over $(x^j,y^j) \in F^j$.

Suppose assumption 5.2-A1 is satisfied. Then, given $(p,\beta) \in R_+^{g+1}$, every optimal solution $(\hat{x}^j,\hat{y}^j)$ of (a) is optimal with respect to (b) with the same p and δ such that $p'\hat{y}^j = \delta.\beta$.

Proposition 5.2.2. and 5.2.3.: Consider, for each $i = 1,2,\dots,m$:

(a) $\sup \varphi_i(z^i,w^i)$, over $(z^i,w^i) \in C^i$, s.t. $p'z^i-p'w^i \leqq \gamma_i$.

(b) $\inf p'z^i-p'w^i$, over $(z^i,w^i) \in C^i$, s.t. $\varphi_i(z^i,w^i) \geqq \alpha_i$.

Then, under 3.3-A4, the following properties hold:

(5.2.2): If, for some $(p,\gamma_i) \in R_+^{g+1}$, $(\hat{z}^i,\hat{w}^i)$ is optimal w.r.t. (a), then $(\hat{z}^i,\hat{w}^i)$ is optimal w.r.t. (b) with the same p and $\alpha_i := \varphi_i(\hat{z}^i,\hat{w}^i)$.

(5.2.3): Let (p,γ_i) be such that (a) possesses an optimal solution. Then every optimal solution of (b), with $\alpha_i :=$ sup in (a) and the same p, is optimal w.r.t. (a).

Exploring this proposition (viz. the proof in section 6) one can deduce:

Theorem 5.2.4.: Under the assumptions 1.5-A1, 3, 4, 5, 1.7-A1, 2, 5, 6, 1.8-A1, 3.3-A1, 2 and 5.2-A1, 2: every path of consumption-supply and production activities, generated by an I.C.E. is weakly efficient. If, in addition, 5.2-A3 is satisfied then such a path is strictly efficient.

6. Proofs.

Proof of 1.8.1. and of 1.8.2.: 1.5-A3 implies the existence of a vector $\tilde{w}$ such that $\Sigma_{i=1}^{m} C^i \subset R_+^g \times [0,\tilde{w}]$. For such a vector $\tilde{w}$:

(1) $\forall (z,w) \in \Sigma_{i=1}^{m} C^i,\ (x,y) \in \Sigma_{j=1}^{n} F^j \mid z-w+x-y \leqq 0:\ z+x-y \leqq \tilde{w}$.

Defining $A := \{(a,v) := (z+x,y) \mid (z,w) \in \Sigma_{i=1}^{m} C^i,\ (x,y) \in \Sigma_{j=1}^{n} F^j\}$, we have:

(2) A is closed and convex. (To be deduced from closedness and convexity of the sets C^i and F^j, and from $\{C^i\}_1^m, \{F^j\}_1^n \subset R_+^{2\cdot g}$).

(3) $(0,0) \in A$. (By 1.5-A1 and 1.7-A1).

(4) $A \cap (\{0\} \times R^g)$ is bounded. (By 1.8-A1 and $\{C^i\}_1^m, \{F^j\}_1^n \subset R_+^{2\cdot g}$).

(5) $\exists \alpha,\beta \in R_+^1 : \forall (a,v) \in A : \|v\|_1 \leqq \alpha + \beta . \|a\|_1$. (By 7.3 and by the properties (2), (3), (4)).

Clearly, the definition of A and the properties (1), (5) imply 1.8.1.

Further, for the set A we have:

(6) $\{(a,v) \in A \mid a \leqq v\} = \{(0,0)\}$. (By 1.8-A1, 1.7-A4, and $\{C^i\}_1^m \subset R_+^{2\cdot g}\}$.

For sequences $\{(z_t,w_t)\}_1^\infty \subset \Sigma_{i=1}^{m} C^i$ and $\{(x_t,y_t)\}_1^\infty \subset F^j$:

(7) $\{(a_t,v_t)\}_1^\infty$ defined by $(a_t,v_t) := (z_t+x_t,y_t)$, $t = 1,\ldots$ is a sequence in A.

(8) In addition, if the inequalities $z_t-w_t+x_t-y_{t-1} \leqq 0$, $t = 1,2,\ldots$ are satisfied then $a_t-v_{t-1} \leqq \tilde{w}$, $t = 1,2,\ldots$ with $v_0 := y_0$. (By (1)).

(9) $\{(a_t,v_t)\}_{t=1}^\infty$ is bounded. (By 7.5, (2), (3), (6), (7), and (8)).

Clearly, (9), the definition of $\{(a_t,v_t)\}_{t=1}^\infty$ and non-negativity of the vectors z_t,x_t, imply 1.8.2.

<u>Proof of 3.4.1.:</u> This property is a direct consequence of boundedness of the sets $\bar{C}^i$, $\bar{\omega}^{i,j}$ and of $\pi_i \in]0,1[$, $i = 1,2,\ldots,m$.

<u>Proof of 3.4.2.:</u> Since the series $\{\Sigma_{t=1}^h (\pi_i)^t.(z_t^i,w_t^i,s_t^{i,1},\ldots,s_t^{i,n})\}_{h=1}^\infty$ converges, property 3.4.2. follows from 7.1. and 7.2.

<u>Proof of 3.4.3.:</u> Compactness of $\bar{C}^i$ and continuity of φ_i implies that, for every $\{(z_t^i,w_t^i)\}_{t=1}^\infty \subset \bar{C}^i$, the sequence $\{\varphi_i(z_t^i,w_t^i)\}_{t=1}^\infty$ is bounded, and henceforth (by $\pi_i \in]0,1[$) convergency of $\{\Sigma_{t=1}^h (\pi_i)^t.\varphi_i(z_t^i,w_t^i)\}_{h=1}^\infty$.

<u>Proof of 3.4.4.:</u> Let $\{(z_t^i,w_t^i)\}_{t=1}^\infty \subset \bar{C}^i$. Defining a sequence $\{(\bar{z}_h^i,\bar{w}_h^i)\}_{h=1}^\infty$ by the convex combinations:

$(\bar{z}_h^i,\bar{w}_h^i) := ((1-\pi_i)/(\pi_i-\pi_i^{h+1})).\Sigma_{t=1}^h (\pi_i)^t.(z_t^i,w_t^i)$, $h = 1,2,\ldots$, concavity of φ_i implies, for each $h = 1,2,\ldots$

(1) $\varphi_i(\bar{z}_h^i,\bar{w}_h^i) \geqq ((1-\pi_i)/(1-\pi_i^{h+1})).\Sigma_{t=1}^h (\pi_i)^t.\varphi_i(z_t^i,w_t^i)$.

Defining $\lambda_i := (1-\pi_i)/\pi_i$, we have

(2) $(\bar{z}_h^i,\bar{w}_h^i) \to \lambda_i.(\tilde{z}^i,\tilde{w}^i)$, for $h\to\infty$.

(By compactness of $\bar{C}^i$ and by $\pi_i \in]0,1[$)

(3) $\varphi_i(\bar{z}_h^i, \bar{w}_h^i) \to \varphi_i(\lambda_i . \tilde{z}^i, \lambda_i . \tilde{w}^i)$, for $h \to \infty$.

(By (2) and by continuity of φ_i).

Combining 3.4.3. and the properties (1), (3), one will find 3.4.4.

Proof of 3.4.5.: It should be clear that the sequence $\{(\hat{z}_t^i, w_t^i, \hat{s}_t^{i,1}, \dots, \hat{s}_t^{i,n})\}_{t=1}^{\infty}$, generated by the optimal solution $\{(\hat{z}^i, \hat{w}^i, \hat{s}^{i,1}, \dots, \hat{s}^{i,n})\}$ of the single-period program 3.4-D2, is feasible w.r.t. 3.4-D1. In addition we have:
$\varphi_i(\hat{z}^i, \hat{w}^i) = ((1-\pi_i)/\pi_i) . \Sigma_{t=1}^{\infty} (\pi_i)^t . \varphi_i(\hat{z}_t^i, \hat{w}_t^i)$.
Now, suppose 3.4-D1 possesses a feasible solution $\{(\bar{z}_t^i, \bar{w}_t^i, \{\bar{s}^{i,j}\}_{j=1}^n)\}_{t=1}^{\infty}$ for which the value of the objective function is higher. Then, by the proposition 3.4.1. to 3.4.4., the single period program possesses a feasible solution - say $(\tilde{z}^i, \tilde{w}^i, \{\tilde{s}^{i,j}\}_{j=1}^n)$ - such that $\varphi_i(\tilde{z}^i, \tilde{w}^i) > \varphi_i(\hat{z}^i, \hat{w}^i)$.
However, this contradicts optimality of $(\hat{z}^i, \hat{w}^i, \{\hat{s}^{i,j}\}_{j=1}^n)$.

Proof of 4.2.6.: For each $j = 1,2,\dots,n$, we distinguish three cases:

(1) $\Sigma_{i=1}^m \hat{s}^{i,j} = 0$, (2) $\Sigma_{i=1}^m \hat{s}^{i,j} > 0$, $\hat{p}'\hat{y}^j = 0$, (3) $\Sigma_{i=1}^m \hat{s}^{i,j} > 0$, $\hat{p}'\hat{x}^j > 0$.
In case (1), non-negativity of $\hat{p}$ and $\hat{x}$ implies $\hat{p}'\hat{x}^j = \Sigma_{i=1}^m \hat{s}^{i,j}$. In case (2), we have $\hat{d}^j = 0$, implying (by 4.2.3.) $\hat{s}^{i,j} = 0$, $i = 1,2,\dots,m$.
Clearly, this contradicts the assumption: $\Sigma_{i=1}^m \hat{s}^{i,j} > 0$. In case (3), proposition 2.3.5. (note: the artificial bound $\bar{\alpha}$ in 2.3.5. is chosen large enough) implies $\hat{p}'\hat{x}^j = \Sigma_{i=1}^m \hat{s}^{i,j}$.

Proof of 4.3.1.: Let $\underset{\sim}{w}^i$ be a supply vector as described in 4.3-A1, and let $\{(\underset{\sim}{x}^j, \underset{\sim}{y}^j)\}_1^n$ be the corresponding production vectors. Then, defining $u^i := \Sigma_{j=1}^n (\underset{\sim}{y}^j - \rho_j . \underset{\sim}{x}^j) + \underset{\sim}{w}^i$, and $\nu_i := \min(u_1^i, u_2^i, \dots, u_n^i)$ (Note $\nu_i > 0$ by 4.3-A1), we have for every $p \in R_+^g$:

$\Sigma_{j=1}^{n} p'(\underset{\sim}{y}^j - \rho_j . \underset{\sim}{x}^j) + p'\underset{\sim}{w}^i \geqq \nu^i . \|p\|_1$.

Since, by consistency of $(p,\{d^j\}_1^n)$: $p'\underset{\sim}{y}^j \leqq d^j . p'\underset{\sim}{x}^j \leqq \rho_j . p'\underset{\sim}{x}^j$, $j = 1,2,\ldots,n$ (viz. 4.2.4.), this implies: $p'\underset{\sim}{w}^i \geqq \nu^i . \|p\|_1$.

Proof of 5.1.3.: Let $(\hat{p},\{\hat{d}^j\}_1^n, \hat{S},\{(\hat{z}^i,\hat{w}^i)\}_1^m,\{(\hat{x}^j,\hat{y}^j)\}_1^n)$ be an I.C.E.
Let $\{(\{(\hat{z}_t^i,\hat{w}_t^i)\}_{i=1}^m,\{(\hat{x}_t^j,\hat{y}_t^j)\}_{j=1}^n)\}_{t=1}^{\infty}$ be an invariant path generated by this I.C.E. Then:

(1) Each $(\hat{z}_t^i,\hat{w}_t^i)$ is optimal w.r.t. max. problem (a) in 5.2.2., with $p := \hat{p}$ and $\gamma_i := \Sigma_{j=1}^n \hat{s}^{i,j}$. (Bij equilibrium condition 4.1-a).

(2) Each $(\hat{z}_t^i,\hat{w}_t^i)$ is optimal w.r.t. min. problem (b) in 5.2.2., with $p := \hat{p}$ and $\alpha_i := \varphi_i(\hat{z}^i,\hat{w}^i)$. (By 5.2.2., property (1), and 3.3-A4).

(3) Each $(\hat{x}_t^j,\hat{y}_t^j)$ is optimal w.r.t. max. problem (a) in 5.2.1., with $p := \hat{p}$ and $\beta := \Sigma_{i=1}^m \hat{s}^{i,j}$ (By equilibrium condition 4.1-a).

(4) Each $(\hat{x}_t^j,\hat{y}_t^j)$ is optimal w.r.t. min. problem (b) in 5.2.1., with $p := \hat{p}$ and $\delta := \hat{d}^j$. (By proposition 5.2.1. and property (3)).

(5) Each $(\hat{x}_t^j,\hat{y}_t^j)$ is optimal w.r.t. min. problem (b) in 5.2.1., with $p := \hat{p}$ and $\delta := \rho^*$, ρ^* such that $\rho_j = \rho^*$, $j = 1,2,\ldots,n$. (viz. 5.2-A1). (By (4) and the fact that $\hat{d}^j < \rho_j$ implies $\hat{s}^{i,j} = 0$, $i = 1,2,\ldots,m$, and so $\hat{p}'\hat{y}^j = 0$, as well. Viz.: 4.2.3. and 4.2.4.).

(6) Defining $\hat{y}_0^j := \hat{y}^j$, $j = 1,2,\ldots,n$, we have, by equilibrium condition 4.1-d: $\Sigma_{i=1}^m (\hat{z}_t^i - \hat{w}_t^i) + \Sigma_{j=1}^n (\hat{x}_t^j - \hat{y}_{t-1}^j) = 0$, $t = 1,2,\ldots$

Defining a set $Q \subset R^g$ by:

$$Q := \Big\{ q \in R^g \mid \exists \{(z_t^i, w_t^i)\}_{t=1}^{\infty} \subset \bar{C}^i,\ i = 1,2,\dots,m,$$

$$\{(x_t^j, y_t^j)\}_{t=1}^{\infty} \subset \bar{F}^j,\ j = 1,2,\dots,n:$$

$$\varphi_i(z_t^i, w_t^i) \geqq \varphi_i(\hat{z}_t^i, \hat{w}_t^i),\ i = 1,2,\dots,m,\ t = 1,2,\dots$$

$$q = \Sigma_{t=1}^{\infty} (1/\rho^*)^t.(\Sigma_{i=1}^m (z_t^i - w_t^i) + \Sigma_{j=1}^n (x_t^j - (1/\rho^*).y_t^j)) \Big\},$$

the properties (2) and (3) imply that

$\hat{q} := \Sigma_{t=1}^{\infty}(1/\rho^*)^t.(\Sigma_{i=1}^m (\hat{x}_t^i - \hat{w}_t^i) + \Sigma_{j=1}^n (\hat{x}_t^j - (1/\rho^*).\hat{y}_t^j))$ is optimal w.r.t. min. $\hat{p}'q$, over $q \in Q$.

Further, by (4), we have $\hat{p}'\hat{q} = (1/\rho^*).\hat{p}'\Sigma_{j=1}^n \hat{y}_0^j$, implying, by optimality of $\hat{q}$ w.r.t. min $\hat{p}'q$, over $q \in Q$:

(7) $\forall\ q \in Q$: $\hat{p}'q \geqq (1/\rho^*).\hat{p}'\Sigma_{j=1}^n \hat{y}_0^j$.

Now, let $\{((\{(\bar{z}_t^i, \bar{w}_t^i)\}_{i=1}^m, \{(\bar{x}_t^j, \bar{y}_t^j)\}_{j=1}^n)\}_{t=1}^{\infty}$ a feasible path which starts from the same initial vectors $y_0^j := \hat{y}^j$, $j = 1,2,\dots,n$, and such that $\varphi_i(\bar{z}_t^i, \bar{w}_t^i) \geqq \varphi_i(\hat{z}_t^i, \hat{w}_t^i)$, $i = 1,2,\dots,m$, $t = 1,2,\dots$ Since this path is feasible, theorem 1.8.2. (based on the assumptions 1.5-A1, 3, 4, 5, 1.7-A1, 4, 5, 6, 1.8-A1) implies that is bounded; i.e.:

$\{(\bar{z}_t^i, \bar{w}_t^i)\}_{t=1}^{\infty} \subset \bar{C}^i$, $i = 1,2,\dots,m$, $\{(\bar{x}_t^j, \bar{y}_t^j)\}_{t=1}^{\infty} \subset \bar{F}^j$, $j = 1,2,\dots,n$.

(Provided the bounds appearing in the definition of $\bar{C}^i$ and $\bar{F}^j$, see 1.8-D1, 2 are chosen large enough). Thus, defining

$\bar{q} := \Sigma_{t=1}^{\infty} (1/\rho^*)^t.(\Sigma_{i=1}^m (\bar{z}_t^i - \bar{w}_t^i) + \Sigma_{j=1}^n (\bar{x}_t^j - (1/\rho^*).\bar{y}_t^j))$, we have

(8) $\bar{q} \in Q$.

(9) $\bar{q} \leqq (1/\rho^*).\hat{p}'\Sigma_{j=1}^n \hat{y}_0^j$. (To be deduced, straight forwards, from feasibility condition 1.8.1. and proposition 7.2.).

Combining (7), (8), and (9):

(10) $\bar{q}$ is optimal w.r.t. min $\hat{p}'q$, over $q \in Q$.

(11) Each $(\bar{z}^i_t, \bar{w}^i_t)$ is optimal w.r.t. min. problem (b) in 5.2.2., with $p := \hat{p}$ and $\alpha_i := \varphi_i(\hat{z}^i, \hat{w}^i)$. (By (10) and the definition of Q).

(12) Each $(\bar{z}^i_t, \bar{w}^i_t)$ is optimal w.r.t. max. problem (a) in 5.2.2., with $p := \hat{p}$ and $\gamma_i := \Sigma^n_{j=1} \hat{s}^{i,j}$. By (11), and proposition 5.2.3. Note: by (1) and (2), and by $\alpha_i := \varphi_i(\hat{z}^i_t, \hat{w}^i_t)$ in (11), γ_i has to be defined: $\gamma_i := \Sigma^n_{j=1} \hat{s}^{i,j}$)

(13) $\varphi_i(\bar{z}^i_t, \bar{w}^i_t) = \varphi_i(\hat{z}^i_t, \hat{w}^i_t)$, $i = 1,2,\dots,m$, $t = 1,2,\dots$ (By (12) and by (1)).

Clearly, the supposition concerning the alternative feasible path: $\varphi_i(\bar{z}^i_t, \bar{w}^i_t) \geqq \varphi_i(\hat{z}^i_t, \hat{w}^i_t)$, $t = 1,2,\dots$, implies equality. (viz. 13). Hence we may conclude that the path generated by the I.C.E. is weakly efficient (viz. efficiency condition 5.2-b).

In order to prove the second part of 5.2.4., concerning strict efficiency, we observe that assumption 5.2-A3 and the definition of the numbers $\{\rho_j\}^n_1$ (viz. 4.2-D1) implies: $\pi_i = 1/\rho^*$, $i = 1,2,\dots,m$, where ρ^* such that $\rho_j = \rho^*$ $j = 1,2,\dots,n$. Putting $\pi^* := 1/\rho^*$, we replace the set Q by:

$$Q^* := \Big\{ q \in R^g_+ \mid \exists \{(z^i_t, w^i_t)\}^\infty_{t=1} \in \bar{C}^i,\ i = 1,2,\dots,m,$$

$$\{(x^j_t, y^j_t)\}^\infty_{t=1} \in \bar{F}^j,\ j = 1,2,\dots,n:$$

$$\Sigma^\infty_{t=1} (\pi^*)^t . \varphi_i(z^i_t, w^i_t) \geqq \Sigma^\infty_{t=1} (\pi^*)^t . \varphi_i(\hat{z}^i_t, \hat{w}^i_t),\ i = 1,2,\dots,m$$

$$q = \Sigma^\infty_{t=1} (\pi^*)^t . \left(\Sigma^m_{i=1} (z^i_t - w^i_t) + \Sigma^n_{j=1} (x^j_t - \pi^* . y^j_t)\right) \Big\}.$$

Further, strict efficiency can be deduced in a similar manner as weak efficiency.

7. Appendix.

Proposition 7.1.: Let $Q \in R^k$ be a closed convex set, and let $\{q_t\}_1^\infty$ be a sequence in Q. Then, for every $\mu \in]0,1[$ such that $\{\Sigma_{t=1}^h \mu^t.q_t\}_{h=1}^\infty$ converges: $((1-\mu)/\mu).\ \Sigma_{t=1}^\infty \mu^t.q_t \in Q$.

Proposition 7.2.: Let $\{(u_t,v_t)\}_{t=1}^\infty \subset R^k \times R_+^k$ be a sequence which satisfies, for some $v_0 \in R_+^k$ and some $w \in R^k$: $u_t - v_{t-1} \leqq w$, $t = 1,2,\ldots$
Then, for every $\mu \in]0,1[$ such that $\{\Sigma_{t=1}^h \mu^t.(u_t-\mu.v_t)\}_{h=1}^\infty$ converges: $\Sigma_{t=1}^\infty \mu^t.(u_t-\mu.v_t) \leqq \mu.v_0 + (\mu/(1-\mu)).w$.

Proposition 7.3.: Let $W \subset R^{2.k}$ be a closed convex set containing the origin (0,0), and such that $W \cap (\{0\} \times R^k)$ is bounded. Then numbers $\alpha,\beta \geqq 0$ exist such that, for every $(w,v) \in W$: $\|v\|_1 \leqq \alpha+\beta\ .\ \|w\|_1$.

Proposition 7.4.: Let $U \subset R^{2.k}$ be a closed convex set containing the origin (0,0), and such that $\{(u,v) \in U \mid u \leqq v\} = \{(0,0)\}$. Then numbers $\alpha,\beta \geqq 0$, $\gamma > 1$ exist such that the relations $(u,v) \in U$, $u-\gamma.v \leqq w$ imply: $\|v\|_1 \leqq \alpha+\beta.\|w\|_1$.

Proposition 7.5.: For a set U as mentioned in 7.4., numbers $\gamma,\delta \geqq 0$ exist such that every sequence $\{(u_t,v_t)\}_{t=1}^\infty \subset U$ which satisfies, for some $\tilde{v}_0 \in R_+^k$ and some $w \in R^k$, $u_1 \leqq \tilde{v}_0$, $u_{t+1}-v_t \leqq w$, $t = 1,2,\ldots$, also satisfies

$\|(u_t,v_t)\|_1 \leqq \gamma+\delta\ .\|(v_0,w)\|_1$, $t = 1,2,\ldots$

Proof of 7.1.: Suppose $\{q_t\}_{t=1}^{\infty}$ is a sequence in Q, and suppose $\mu \in]0,1[$ is a number such that $\{\Sigma_{t=1}^{h} \mu^t.q_t\}_{h=1}^{\infty}$ converges. Defining a sequence $\{\bar{q}_h\}_{h=1}^{\infty}$ by the convex combinations:

$\bar{q}_h := ((1-\mu)/(\mu-\mu^{h+1})).\ \Sigma_{t=1}^{h} \mu^t.q_t,\ h = 1,2,\ldots,$

we may conclude (by convexity of Q): $\bar{q}_h \in Q$, $h = 1,2,\ldots$ In addition, convergency of $\{\Sigma_{t=1}^{h} \mu^h.q_t\}_{h=1}^{\infty}$ and $\mu \in]0,1[$ implies convergency of $\{\bar{q}_h\}_{h=1}^{\infty}$.

Clearly, by closedness of Q, these relations imply $((1-\mu)/\mu).\Sigma_{t=1}^{\infty} \mu^t.q_t \in Q$.

Proof of 7.2.: $u_t - v_{t-1} \leqq w$, $v_t \geqq 0$, $t = 1,2,\ldots$ implies, for every $\mu > 0$:

$\Sigma_{t=1}^{h} \mu^t.(u_t - \mu.v_t) \leqq \mu.v_0 + ((\mu-\mu^{h+1})/(1-\mu)).w,\ h = 1,2,\ldots$

Clearly, $\mu \in]0,1[$ and convergency of $\{\Sigma_{t=1}^{h} \mu^t.(u_t-\mu.v_t)\}_{h=1}^{\infty}$ implies:

$\Sigma_{t=1}^{\infty} \mu^t.(u_t-\mu.v_t) \leqq \mu.v_0 + (\mu/(1-\mu)).w.$

Proof of 7.3.: Following Rockafellar [7], we define the recession cone, rec (A), of a set $A \subset R^k$ by: rec $(A) := \{a \in R^k \mid \forall\, b \in A,\ \lambda \geqq 0 : b + \lambda.a \in A\}$.

Now, defining a set $V := \{(w,v) \in R^{2.k} \mid \|w\|_1 \leqq 1\}$, we have with respect to a set W as assumed in 7.3.:

(1) rec $(W \cap V)$ = rec $(W) \cap$ rec (V). (By closedness and convexity of W and V. See Rockafellar 8.3.3.).

(2) rec $(V) \subset \{0\} \times R^k$. (By the definition of V).

(3) rec $(W) \cap (\{0\} \times R^k) = \{(0,0)\}$. (By boundedness of the set $W \cap (\{0\} \times R^k)$.)

(4) rec $(W \cap V) = \{(0,0)\}$. (By (1), (2), and (3).)

(5) $(W \cap V)$ is bounded. (By (4), and by closedness and convexity of the set $W \cap V$. See Rockafellar 8.4.)

(6) $\exists\ \delta > 0 : \forall\ (w,v) \in W \cap V : \|v\|_1 \leqq \delta$. (By (5).)

(7) $\forall\ (w,v) \in W : (1 + \|w\|_1)^{-1}.(w,v) \in W \cap V$. (By the definition of set V, convexity of V, W, and by $(0,0) \in W \cap V$.)

(8) $\exists\ \delta > 0 : \forall\ (w,v) \in W : \|v\|_1 \leqq \delta.(1 + \|w\|_1)$. (By (6), (7), and by the definition of set V.)

Proof of 7.4.: Defining $T := \{(u,v) \in R^{2.k} \mid u \leqq v\}$, we have, for a set U as assumed in 7.5., the following properties:

(1) $\text{rec}\ (U \cap T) = \{(0,0)\}$. (By closedness and convexity of U,T and by boundedness of $U \cap T$. See Rockafellar 8.4.)

(2) $\text{rec}\ (U) \cap T = \{(0,0)\}$. (By closedness and convexity of U,T, by (1), and by $\text{rec}\ (T) = T$. See Rockafellar 8.3.3.)

(3) $\exists\ \gamma > 1 : \{(u,v) \in R^{2.k} \mid u \leqq \gamma.v\} \cap \text{rec}\ (U) = \{(0,0)\}$. (By (2), and by closedness of rec (U). See Rockafellar 8.2.)

(4) $\exists\ \gamma > 1 : \text{rec}\ (\{(u,v) \in R^{2.k} \mid u \leqq \gamma.v\} \cap U) = \{(0,0)\}$. (By (4), and by closedness and convexity of the sets U and $\{(u,v) \in R^{2.k} \mid u \leqq \gamma.v\}$. See Rockafellar 8.3.3.)

Now, defining, for a $\gamma > 1$ such that (4) holds, a set W by
$W := \{(w,v) \in R^{2.k} \mid \exists\ u \in R^{k} : (u,v) \in U,\ u-\gamma.v \leqq w\}$, we have:

(5) W is closed, convex, and contains the origin. (By the assumptions concerning U.)

(6) $W \cap (\{0\} \times R^k)$ is bounded. (By (4), (5), and by the definition of W. See Rockafellar 8.4.)

(7) $\exists\ \gamma > 1,\ \delta \geqq 0$: $\forall\ u,v,w \in R^k \mid (u,v) \in U,\ u-\gamma.v \leqq w$: $\|v\|_1 \leqq \delta.\|w\|_1 + \delta$. (By 7.3., by (6), and by the definition of W.)

Proof of 7.5.: Let $\alpha,\beta \geqq 0$, $\gamma > 0$, be the numbers as mentioned in 7.4. Let, for some $v_0 \in R^k_+$ and some $w \in R^k$, $\{(u_t,v_t)\}_{t=1}^{\infty} \subset U$ be sequence which satisfies $u_t - v_{t-1} \leqq w$, $t = 1,2,\ldots$ Defining a sequence $\{(\bar{u}_h,\bar{v}_h)\}_{h=1}^{\infty}$ by the convex combinations: $(\bar{u}_h,\bar{v}_h) := ((\gamma-1)/(\gamma-\gamma^{1-h})).\Sigma_{t=1}^{h}\ \gamma^{t-h}.(u_t,v_t)$, $h = 1,2,\ldots,$ we have:

(1) $(\bar{u}_h,\bar{v}_h) \in U$, $h = 1,2,\ldots$ (By convexity of U.)

(2) $\bar{u}_h - \gamma.\bar{v}_h \leqq w + (\gamma^{-h}(\gamma-1)/(\gamma-\gamma^{1-h})).v_0$, $h = 1,2,\ldots$ (By $u_t - v_{t-1} \leqq w$, $t = 1,2,\ldots,$ and by $v_t \geqq 0$, $t = 1,2,\ldots$)

Since the coefficient $\gamma^{-h}(\gamma-1)/(\gamma-\gamma^{1-h})$ is non-increasing with respect to h (implied by $\delta > 1$), property (2) implies the existence of a vector $\bar{w}$ such that:

(3) $\bar{u}_h - \gamma.\bar{v}_h \leqq \bar{w}$, $h = 1,2,\ldots$

By virtue of 7.4., the relations (1) and (3) imply:

(4) $\|\bar{v}_h\|_1 \leqq \alpha + \beta\ .\|\bar{w}\|_1$, $h = 1,2,\ldots,$ where α,β are the numbers indicated by 7.4.

(5) $\|v_h\|_1 \leqq \alpha+\beta.\|\bar{w}\|_1$, $h = 1,2,\ldots$ (By (4), by $v_t \geqq 0$, $t = 1,2,\ldots,$ and by the definition of $\{(\bar{u}_h,\bar{v}_h)\}_{h=1}^{\infty}$.)

(6) $\|u_n\|_1 \leqq \|w\|_1 + \alpha + \beta . \|\bar{w}\|_1$, $h = 1,2,\ldots$ (By (5), and by

$u_t \leqq w + v_t$, $u_t \geqq 0$, $t = 1,2,\ldots$).

Clearly, the inequalities (5), (6), and the definition of $\bar{w}$, imply 7.5

List of symbols.

R^n, n-dimensional vectorspace

$R^n_+ := \{x \in R^n \mid x_i \geqq 0,\ i = 1,2,\ldots,n\}$, the non-negative orthant.

$\|.\|_1$, the l_1-norm, for $x \in R^n$, defined by $\|x\|_1 := \Sigma_{i=1}^n |x_i|$.

$\delta.x$ scalar-vector multiplication.

$x'y$, the inner product of a pair of finite dimensional vectors.

$[x,y] := \{z \in R^n \mid z \geqq x,\ z \leqq y\}$, the n-dim. closed interval.

$]x,y] := \{z \in R^n \mid z > x,\ z \leqq y\}$

$[x,y[:= \{z \in R^n \mid z \geqq x,\ z < y\}$

$]x,y[:= \{z \in R^n \mid z > x,\ z < y\}$

$[\alpha,\beta]^n := \{z \in R^n \mid z_i \in [\alpha,\beta],\ i = 1,2,\ldots,n\}$, α,β being scalars.

The sum of sets $A^1, A^2, \ldots, A^k$ in R^n:

$\Sigma_{i=1}^k A^i := \{x \in R^n \mid \exists\{x^i\}_1^k \subset R^n : x^i \in A^i,\ i = 1,2,\ldots,k,\ x = \Sigma_{i=1}^k x^i\}$.

$C : X \to Y$, single-valued function from X to Y.

$C : X \twoheadrightarrow Y$, set-valued function from X to Y.

Graph of $C : X \twoheadrightarrow Y$: graph $(X;C) := \{(x,y) \in X \times Y \mid y \in C(x)\}$.

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